

The University of Chicago

RELATIONS BETWEEN THE METRIC AND
PROJECTIVE THEORIES OF
SPACE CURVES

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL
OF SCIENCE IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

BY

THOMAS McNIDER SIMPSON, JR.

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INTRODUCTION.

The essential distinction between metric and projective properties in geometry lies in the transformations under which they are invariant.¹ As the group of motions is a subgroup of the general projective group, metric properties are defined by the invariants—algebraic, differential, or integral—of this subgroup. Figures metrically equivalent are projectively equivalent, but projectively equivalent figures may admit of further classification according to their metric properties.

It would seem to be quite evident then that the metric properties of figures might be studied by considering first the invariants of the projective theory, specializing the independent variables, and properly restricting the admissible transformations.

Halphen was the first to obtain the general forms for the projective differential invariants of space curves. He also exhibited the relation between these quantities and the invariants of a homogeneous linear differential equation of the fourth order.²

Interpreting the four linearly independent solutions of such an equation as the homogeneous coordinates of a point in space, and allowing the independent parameter to vary, the point so defined describes a space curve. Since any four linearly independent combinations of these form again a fundamental system of solutions, and since the relation between such sets is of the same form as the projective transformation of coordinates in space, it is clear that the properties defined by the invariants of the differential equation in question will be properties not of some one curve but of a family of projectively equivalent curves.

Wilczynski has presented the invariants in more compact form and has developed the theory for plane curves and for curves and surfaces in space in much detail.³ He has also carried through the work of relating the metric to the projective theory of plane curves,⁴ and similar problems have been studied for other configurations by some of his students.⁵

¹ Klein, *Comparative Review of Recent Researches in Geometry*. (Translation of the Erlanger Program, 1872.) *Bulletin of the New York Mathematical Society*, II (1892-3), pp. 217-220.

² Halphen, *Sur les invariants différentiels des courbes gauches*, *Journal de l'école polytechnique*, Cah. 47 (1880), 1-102.

Halphen, *Sur les invariants des équations différentielles linéaires du quatrième ordre*, *Acta Mathematica*, 3 (1883-4), 325-380.

³ Wilczynski, *Projective Differential Geometry of Curves and Surfaces*. Hereinafter referred to as *Proj. Diff. Geom.*

⁴ Presented in lectures at the University of Chicago in the spring of 1916. Unpublished.

⁵ Morrison, *On the Relation between some important Notions of Projective and Metric Differential Geometry*, *American Journal of Mathematics*, XXXIX (1917), pp. 199-220.

It is at Professor Wilczynski's suggestion and with the invaluable aid of his direction that this study in the theory of space curves has been undertaken.

In the projective theory the invariants are expressed in terms of a perfectly general parameter; for the metric theory we shall find it convenient to use the arc length, which is the simplest integral invariant, as the independent parameter and to express the projective invariants in terms of the radii of curvature and of torsion and their derivatives with respect to the arc length s , all of these quantities being metric differential invariants.

We shall assume that the given curve is defined by the intrinsic equations¹ in the explicit form,

$$r = r(s); \quad \tau = \tau(s);$$

where r and τ denote the radii of curvature and of torsion respectively; and we shall assume that these functions are expansible in convergent power series in s . We shall confine our attention to those points of the curve which are regular in the sense that r and τ are both finite and not equal to zero.

The fourth order homogeneous linear differential equation which is so important in the projective theory has the coordinates of the points of the curve considered as functions of s as linearly independent solutions. When this equation has been found, the projective invariants may be determined.

Properties of the curve which are expressible in terms of the projective invariants alone are projective properties; metric properties will be expressed by functions of r , τ , and their derivatives, which need not of course be projective invariants.

We shall set up the equations of transformation between the coordinate systems most convenient for the study of projective properties and those most convenient for the investigation of metric properties. We shall then consider a considerable number of different osculating surfaces and curves. Whenever one of these osculants is connected with the given curve by properties which are essentially of a metric character, our method of derivation will exhibit this fact, but whenever the connection between the curve and the osculant is essentially projective, the corresponding equations and properties will be obtained by transformation from the equations of projective differential geometry.

The actual determination, in a given case, of any osculant depends upon the knowledge of certain derivatives of the radii of curvature and torsion. Thus these osculants may be used as geometric images of these derivatives. We shall actually carry out the geometrical interpretation of these derivatives in this sense more completely than has been done heretofore. Finally we shall apply our results to certain special classes of curves.

Kingston, Metric Properties of Nets of Plane Curves. *American Journal of Mathematics*, XXXVIII (1916), pp. 407-430.

Reaves, Metric Properties of Fleenodes on Ruled Surfaces. *Giornali di Matematiche di Battaglini*, Vol. 55 (1917).

¹ Cesaro, *Vorlesungen über natürliche Geometrie*, p. 156.

Kommerell, *Allgemeine Theorie der Raumkurven und Flächen*, I, p. 29.

1. THE FUNDAMENTAL EQUATION AND INVARIANTS.

The arc length s being chosen as the independent parameter, and the direction cosines of the tangent, principal normal, and binormal to the given curve at the point (x, y, z) being denoted respectively by $\alpha, \beta, \gamma; l, m, n; \lambda, \mu, \nu$; we have by differentiation and the application of the Frenet-Serret formulas,¹

$$(1) \quad \begin{aligned} x' &= \alpha, & x'' &= \frac{l}{r}, & x''' &= -\left(\frac{\alpha}{r^2} + \frac{r'l}{r^2} + \frac{\lambda}{r\tau}\right), \\ x^{iv} &= \frac{3r'}{r^3} \alpha + \left(\frac{2r'}{r^2\tau} + \frac{\tau'}{r\tau^2}\right) \lambda + \left(\frac{2r'^2}{r^3} - \frac{r''}{r^2} - \frac{1}{r\tau^2} - \frac{1}{r^3}\right) l, \end{aligned}$$

and the analogous formulas for the derivatives of x and y are obtained by cyclic permutation of the three sets of direction cosines.

Interpreting $x, y, z, 1$ as homogeneous Cartesian coordinates, we seek the linear homogeneous differential equation of the fourth order which has $x, y, z, 1$ as linearly independent solutions. The equation is of the form

$$(2) \quad \mathbf{y}^{iv} + 4p_1\mathbf{y}''' + 6p_2\mathbf{y}'' + 4p_3\mathbf{y}' + p_4\mathbf{y} = 0.$$

Since $\mathbf{y} = 1$ is a solution, we have $p_4 = 0$. Since x, y, z , are solutions, we have

$$(3) \quad \begin{aligned} \left(\frac{3r'}{r^3} - \frac{4p_1}{r^2} + 4p_3\right) \alpha + \left(\frac{2r'}{r^2\tau} + \frac{\tau'}{r\tau^2} - \frac{4p_1}{r\tau}\right) \lambda \\ + \left(\frac{2r'^2}{r^3} - \frac{r''}{r^2} - \frac{1}{r\tau^2} - \frac{1}{r^3} - \frac{4p_1r'}{r^2} + \frac{6p_2}{r}\right) l = 0, \end{aligned}$$

and the two corresponding equations obtained by permuting the direction cosines. Since the determinant of the direction cosines is different from zero, the expressions in parentheses are separately equal to zero; whence

$$(4) \quad \begin{aligned} 4p_1 &= \frac{2r'}{r} + \frac{\tau'}{\tau}, \\ 6p_2 &= \frac{1}{r^2} + \frac{1}{\tau^2} + \frac{r'\tau'}{r\tau} + \frac{r''}{r}, \\ 4p_3 &= \frac{1}{r^2} \left(\frac{\tau'}{\tau} - \frac{r'}{r}\right). \end{aligned}$$

The fundamental equation is therefore

$$(5) \quad \mathbf{y}^{iv} + \left(\frac{2r'}{r} + \frac{\tau'}{\tau}\right) \mathbf{y}''' + \left(\frac{1}{r^2} + \frac{1}{\tau^2} + \frac{r'\tau'}{r\tau} + \frac{r''}{r}\right) \mathbf{y}'' + \frac{1}{r^2} \left(\frac{\tau'}{\tau} - \frac{r'}{r}\right) \mathbf{y}' = 0.$$

¹ Frenet, *Journal de Mathematiques*, XVII (1852), 438-9; extract from a thesis presented in 1847 to the faculty of Toulouse.

Serret, *Journal de Mathematiques*, XVI (1851), 194-5.

Any function of the coefficients of (5), invariant under the transformation

$$Y = \Lambda(s)\bar{Y},$$

which leaves the ratios of the homogeneous coordinates unaltered, is called a seminvariant. The simplest of these seminvariants are¹

$$(6) \quad \begin{aligned} P_2 &= p_2 - p_1' - p_1^2, \\ P_3 &= p_3 - p_1'' - 3p_1p_2 + 2p_1^3. \end{aligned}$$

Making use of (4), (6) may be written

$$(7) \quad \begin{aligned} P_2 &= \frac{1}{48r^2\tau^2} [8r^2 + 8\tau^2 - 4rr'\tau\tau' - 16rr''\tau^2 - 12r^2\tau\tau'' + 12r'^2\tau^2 + 9r^2\tau'^2], \\ P_3 &= \frac{1}{32r^3\tau^3} [4r\tau^2\tau' - 16r'\tau^3 - 8r^2r'\tau - 4r^3\tau' - 16r^2r''\tau^3 - 8r^3\tau^2\tau''' + 40rr'r''\tau^3 \\ &\quad + 24r^3\tau\tau'\tau'' - 24r'^3\tau^3 - 15r^3\tau'^3 - 4r^2r''\tau^2\tau' + 4rr'^2\tau^2\tau' + 2r^2r'\tau\tau'^2]. \end{aligned}$$

The simplest function of the coefficients which is invariant under the above transformation combined with an arbitrary transformation upon the independent parameter is²

$$(8) \quad \Theta_3 = P_3 - \frac{3}{2} P_2',$$

which may be written

$$(9) \quad \Theta_3 = \frac{1}{32r^2\tau^3} [4\tau^2\tau' - 8rr'\tau + 12r^2\tau' + 4r^2\tau^2\tau'' - 6r^2\tau\tau'\tau'' + 3r^2\tau'^3 - 2rr'\tau\tau'^2 + 4rr'\tau^2\tau''].$$

If, instead of using intrinsic equations, we define the given curve by the equations

$$(10) \quad y = y(x), \quad z = z(x),$$

and denote by y' , y'' , etc., derivatives with respect to x , we find that the fundamental equation corresponding to (5) has the form

$$(11) \quad Y^{iv} + \frac{y^{iv}z'' - y''z^{iv}}{y''z''' - y'''z''} Y''' + \frac{y'''z^{iv} - y^{iv}z'''}{y''z''' - y'''z''} Y'' = 0,$$

and the invariant Θ_3 becomes

$$(12) \quad \Theta_3 = \frac{1}{32(y''z''' - y'''z'')^3} [4(y^{iv}z'' - y''z^{iv})(y'''z'' - y''z''')^2 + 16(y^{iv}z''' - y'''z^{iv})(y'''z'' - y''z''')^2 - 18(y^{iv}z'' - y''z^{iv})(y^{iv}z'' - y''z^{iv})(y'''z'' - y''z''') - 30(y^{iv}z''' - y'''z^{iv})(y^{iv}z'' - y''z^{iv})(y'''z'' - y''z''') + 15(y^{iv}z'' - y''z^{iv})^3].$$

¹ Proj. Diff. Geom., p. 239.

² Proj. Diff. Geom., p. 240.

This is, except for a numerical factor, the differential invariant of order six found by Halphen;¹ it is identically equal to zero for a curve all of whose tangents belong to the same linear complex. Wilczynski² obtained the same interpretation for the identical vanishing of Θ_3 , by showing that this is the necessary and sufficient condition that the same linear complex osculates the curve at each of its points.

2. THE COORDINATE SYSTEMS AND THEIR RELATIONS.

We shall denote by P_Y the point of the given curve whose homogeneous Cartesian coordinates are

$$(13) \quad Y_1 = x, \quad Y_2 = y, \quad Y_3 = z, \quad Y_4 = 1,$$

all of which are solutions of (5).

The relative semicovariants³ of (5) are given by

$$(14) \quad \begin{aligned} z &= Y' + p_1 Y, \\ \rho &= Y'' + 2p_1 Y' + p_2 Y, \\ \sigma &= Y''' + 3p_1 Y'' + 3p_2 Y' + p_3 Y. \end{aligned}$$

Under the transformation $Y = \Lambda(s)\bar{Y}$, z , ρ , σ will be transformed into $\bar{z}$, $\bar{\rho}$, $\bar{\sigma}$ in accordance with the equations

$$z = \Lambda(s)\bar{z}, \quad \rho = \Lambda(s)\bar{\rho}, \quad \sigma = \Lambda(s)\bar{\sigma}.$$

Substitution of (13) in the right members of (14) gives values z_i , ρ_i , σ_i corresponding to Y_i which we may interpret as homogeneous coordinates of points which we shall denote by P_z , P_ρ , P_σ . Making this substitution and using equations (1), we obtain

$$(15) \quad \begin{aligned} z_1 &= \alpha + p_1 x, \\ z_2 &= \beta + p_1 y, \\ z_3 &= \gamma + p_1 z, \\ z_4 &= p_1, \end{aligned}$$

as the homogeneous coordinates of the point P_z ;

$$(16) \quad \begin{aligned} \rho_1 &= 2p_1\alpha + \frac{1}{r}l + p_2x, \\ \rho_2 &= 2p_1\beta + \frac{1}{r}m + p_2y, \\ \rho_3 &= 2p_1\gamma + \frac{1}{r}n + p_2z, \\ \rho_4 &= p_2, \end{aligned}$$

¹ Halphen, Journal de l'école polytechnique, Cah. 47 (1880), p. 8. Acta Mathematica, Vol. 3 (1883), p. 328.

² Proj. Diff. Geom., p. 254.

³ Proj. Diff. Geom., p. 239.

as the coordinates of P_ρ ; and

$$\begin{aligned}
 \sigma_1 &= \left(3p_2 - \frac{1}{r^2}\right) \alpha + \frac{1}{2r} \left(2p_1 + \frac{\tau'}{\tau}\right) l - \frac{1}{r\tau} \lambda + p_3 x, \\
 \sigma_2 &= \left(\right) \beta + \frac{1}{2r} \left(\right) m - \frac{1}{r\tau} \mu + p_3 y, \\
 \sigma_3 &= \left(\right) \gamma + \frac{1}{2r} \left(\right) n - \frac{1}{r\tau} \nu + p_3 z, \\
 \sigma_4 &= p_3,
 \end{aligned}
 \tag{17}$$

for the coordinates of P_σ .

The points $P_Y, P_Z, P_\rho, P_\sigma$ are the vertices of a tetrahedron. P_Y is the point P of the curve under consideration; (15) shows that P_Z is a point on the tangent to the curve at P , and (16) shows that P_ρ lies in the osculating plane at P . We adopt this tetrahedron as the local tetrahedron of reference for a system of homogeneous coordinates.

We shall denote by x, y, z , the coordinates of P_Y , a point on the given curve, referred to a fixed Cartesian system; by X, Y, Z , the coordinates of a general point referred to the fixed system; and by ξ, η, ζ , the coordinates of the same general point referred to the local trihedral at P . The X -coordinates of the vertices of the local tetrahedron are given by

$$\begin{aligned}
 X_Y &= x, \\
 X_Z &= x + \frac{1}{p_1} \alpha, \\
 X_\rho &= x + \frac{2p_1}{p_2} \alpha + \frac{1}{rp_2} l, \\
 X_\sigma &= x + \frac{3p_2r^2 - 1}{r^2p_3} \alpha + \frac{2p_1\tau + \tau'}{2r\tau p_3} l - \frac{1}{r\tau p_3} \lambda,
 \end{aligned}
 \tag{18}$$

and the other coordinates are given by an obvious permutation of letters. Since the relations between the fixed system and the local trihedral are given by the diagram

	X	Y	Z
ξ	α	β	γ
η	l	m	n
ζ	λ	μ	ν

we find for the coordinates of the local tetrahedron referred to the local trihedral,

$$\begin{aligned}
 P_Y: & 0, & 0, & 0, \\
 P_Z: & \frac{1}{p_1}, & 0, & 0, \\
 P_\rho: & \frac{2p_1}{p_2}, & \frac{1}{rp_2}, & 0, \\
 P_\sigma: & \frac{3p_2r^2 - 1}{r^2p_3}, & \frac{2p_1\tau + \tau'}{2r\tau p_3}, & -\frac{1}{r\tau p_3}.
 \end{aligned}
 \tag{19}$$

If we form a linear combination of the fundamental semicovariants with arbitrary constant coefficients x_1, x_2, x_3, x_4 , we have a new semicovariant,

$$(20) \quad \chi = x_1\gamma + x_2\delta + x_3\rho + x_4\sigma.$$

Substituting successively the four independent solutions of (5) in the semicovariants in the right member of (20), we obtain four values of the semicovariant χ which we may interpret as the homogeneous Cartesian coordinates of a point P_χ referred to the fixed system. Since $P_\gamma, P_\delta, P_\rho, P_\sigma$ are not coplanar, x_1, x_2, x_3, x_4 can be so chosen that P_χ shall be any arbitrary point of space. The unit point of the local tetrahedron $P_\gamma P_\delta P_\rho P_\sigma$ can be so chosen that x_1, x_2, x_3, x_4 shall be the homogeneous coordinates of the point P_χ referred to this local tetrahedron.¹ In fact it is only necessary to choose as the unit point the point whose homogeneous Cartesian coordinates referred to the fixed system are

$$\kappa\chi_i = \gamma_i + \delta_i + \rho_i + \sigma_i \quad (i = 1, 2, 3, 4).$$

Making use of the relations

$$X = \frac{\chi_1}{\chi_4}, \quad Y = \frac{\chi_2}{\chi_4}, \quad Z = \frac{\chi_3}{\chi_4},$$

$$\xi = (X - x)\alpha + (Y - y)\beta + (Z - z)\gamma, \quad \text{etc.,}$$

together with equations (13), (15), (16), (17), we obtain the equations of transformation connecting the local trihedral coordinates ξ, η, ζ and the local tetrahedral coordinates x_1, x_2, x_3, x_4 ,

$$(21) \quad \begin{aligned} \xi &= \frac{x_2 + 2p_1x_3 + \left(3p_2 - \frac{1}{r^2}\right)x_4}{x_1 + p_1x_2 + p_2x_3 + p_3x_4}, \\ \eta &= \frac{\frac{1}{r}x_3 + \frac{2p_1\tau + \tau'}{2r\tau}x_4}{\text{same denominator}}, \\ \zeta &= \frac{-\frac{1}{r\tau}x_4}{\text{same denominator}}. \end{aligned}$$

The inverse transformation is

$$(22) \quad \begin{aligned} \kappa x_1 &= 1 - p_1\xi + r(2p_1^2 - p_2)\eta + \left(2p_1^3r\tau + p_1^2r\tau' - 4p_1p_2r\tau + p_1\frac{\tau}{r} - \frac{1}{2}p_2r\tau' + p_3r\tau\right)\zeta, \\ \kappa x_2 &= \xi - 2p_1r\eta - \left\{\left(2p_1^2r - 3p_2r + \frac{1}{r}\right)\tau + p_1r\tau'\right\}\zeta, \\ \kappa x_3 &= r\eta + \frac{r(2p_1\tau + \tau')}{2}\zeta, \\ \kappa x_4 &= -r\tau\zeta, \end{aligned}$$

κ being merely a proportionality factor.

¹ Proj. Diff. Geom., pp. 61, 244.

We shall have occasion to make use of line coordinates referred to the local trihedral and to the local tetrahedron. We proceed to develop the relations between them. Let

$$\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{42}, \omega_{34},$$

be the homogeneous coordinates of a line referred to the local tetrahedron; and let

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{42}, \pi_{34},$$

be the homogeneous coordinates of the same line referred to the local trihedral at P .

If (ξ, η, ζ) and (ξ', η', ζ') be any two points on a line, its coordinates are¹

$$(23) \quad \begin{aligned} \kappa\pi_{12} &= \begin{vmatrix} \xi & \eta \\ \xi' & \eta' \end{vmatrix}, & \kappa\pi_{13} &= \begin{vmatrix} \xi & \zeta \\ \xi' & \zeta' \end{vmatrix}, & \kappa\pi_{14} &= \begin{vmatrix} \xi & 1 \\ \xi' & 1 \end{vmatrix}, \\ \kappa\pi_{23} &= \begin{vmatrix} \eta & \zeta \\ \eta' & \zeta' \end{vmatrix}, & \kappa\pi_{42} &= \begin{vmatrix} 1 & \eta \\ 1 & \eta' \end{vmatrix}, & \kappa\pi_{34} &= \begin{vmatrix} \zeta & 1 \\ \zeta' & 1 \end{vmatrix}. \end{aligned}$$

Similarly, referred to the local tetrahedron,²

$$(24) \quad \kappa\omega_{ij} = \begin{vmatrix} x_i & x_j \\ x_i' & x_j' \end{vmatrix}.$$

In general, if homogeneous point coordinates referred to two systems are connected by the relations

$$(25) \quad \kappa x_i = \sum_{h=1}^4 a_{ih} x_h' \quad (i = 1, \dots, 4),$$

the corresponding line coordinates are connected by the relations³

$$(26) \quad \kappa\pi_{ik} = \sum_{h,j=1}^4 (a_{ih}a_{kj} - a_{kh}a_{ij})\pi_{hj}'.$$

Applying equation (26) to (21) and (22), and introducing the notations

$$(27) \quad \begin{aligned} q_1 &= \frac{1}{2}(2p_1\tau + \tau'), \\ q_2 &= 3p_2 - \frac{1}{r^2}, \end{aligned}$$

we obtain the desired transformation,

$$(28) \quad \begin{aligned} \kappa\pi_{12} &= r\tau\omega_{23} - q_1r\omega_{42} & + r(2p_1q_1 - \tau q_2)\omega_{34}, \\ \kappa\pi_{13} &= r\omega_{42} & - 2p_1r\omega_{34}, \\ \kappa\pi_{14} &= -r^2\tau\omega_{12} - 2p_1r^2\tau\omega_{13} - q_2r^2\tau\omega_{14} \\ &+ (p_2 - 2p_1^2)r^2\tau\omega_{23} + (p_1q_2 - p_3)r^2\tau\omega_{42} + (2p_1p_3 - p_2q_2)r^2\tau\omega_{34}, \\ \kappa\pi_{23} &= & - \omega_{34}, \\ \kappa\pi_{42} &= r\tau\omega_{13} + q_1r\omega_{14} & + p_1r\tau\omega_{23} & - p_1q_1r\omega_{42} & + (p_2q_1 - p_3r\tau)\omega_{34}, \\ \kappa\pi_{34} &= r\omega_{14} & - p_1r\omega_{42} & + p_2r\omega_{34}, \end{aligned}$$

¹ Proj. Diff. Geom., p. 158.

² Proj. Diff. Geom., p. 50.

³ Zindler, Liniengeometrie, I, p. 111.

the inverse of which is

$$\begin{aligned}
 \kappa\omega_{12} &= p_2 r \pi_{12} + r(p_2 q_1 - p_3 \tau) \pi_{13} - \pi_{14} \\
 &\quad + r^2 \tau (2p_1 p_3 - p_2 q_2) \pi_{23} - 2p_1 r \pi_{42} + r(2p_1 q_1 - q_2 \tau) \pi_{34}, \\
 \kappa\omega_{13} &= -p_1 r \pi_{12} - p_1 q_1 r \pi_{13} \\
 &\quad + r^2 \tau (p_1 q_2 - p_3) \pi_{23} + r \pi_{42} - q_1 r \pi_{34}, \\
 (29) \quad \kappa\omega_{14} &= p_1 r \tau \pi_{13} \\
 &\quad - r^2 \tau (2p_1^2 - p_2) \pi_{23} + r \tau \pi_{34}, \\
 \kappa\omega_{23} &= r \pi_{12} + q_1 r \pi_{13} \\
 &\quad - r^2 \tau q_2 \pi_{23}, \\
 \kappa\omega_{42} &= r \tau \pi_{13} \\
 &\quad - 2r^2 \tau p_1 \pi_{23}, \\
 \kappa\omega_{34} &= \\
 &\quad - r^2 \tau \pi_{23}.
 \end{aligned}$$

In a purely projective study of curves, we may reduce the fundamental equation to a canonical form in which $P_2 = 0$, by transforming the independent variable. It will be useful to have the relation between the homogeneous coordinates of a point referred to the tetrahedron corresponding to the independent parameter s and to that which corresponds to the case just indicated.

If $\psi(s)$ denote the parameter to be used instead of s in order that P_2 shall reduce to zero, and θ denote ψ''/ψ' , the semicovariants z, ρ, σ will be replaced by $\bar{z}, \bar{\rho}, \bar{\sigma}$, where¹

$$\begin{aligned}
 \bar{z} &= \frac{1}{\psi'} \left(z + \frac{3}{2} \theta y \right), \\
 (30) \quad \bar{\rho} &= \frac{1}{\psi'^2} \left[\rho + 2\theta z + \left(\frac{4}{5} P_2 + \frac{3}{2} \theta^2 \right) y \right], \\
 \bar{\sigma} &= \frac{1}{\psi'^3} \left[\sigma + \frac{3}{2} \theta \rho + \left(\frac{6}{5} P_2 + \frac{3}{2} \theta^2 \right) z + \frac{3}{20} (2P_2' + 8\theta P_2 + 5\theta^3) y \right].
 \end{aligned}$$

Let $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$, and x_1, x_2, x_3, x_4 , be the homogeneous coordinates of the same point referred to the two different local coordinate systems. Then we shall have, by equating the corresponding right members of (20) in the two cases,

$$(31) \quad \bar{x}_1 \bar{y}_i + \bar{x}_2 \bar{z}_i + \bar{x}_3 \bar{\rho}_i + \bar{x}_4 \bar{\sigma}_i = \kappa [x_1 y_i + x_2 z_i + x_3 \rho_i + x_4 \sigma_i] \quad (i = 1, 2, 3, 4).$$

Substituting in the left member from (30) and equating the coefficients of $y_i, z_i, \rho_i, \sigma_i$ respectively, we arrive at the equations of transformation,

$$\begin{aligned}
 \kappa x_1 &= \bar{x}_1 + \frac{3}{2} \frac{\theta}{\psi'} \bar{x}_2 + \frac{1}{\psi'^2} \left(\frac{4}{5} P_2 + \frac{3}{2} \theta^2 \right) \bar{x}_3 + \frac{3}{20 \psi'^3} (2P_2' + 8\theta P_2 + 5\theta^3) \bar{x}_4, \\
 \kappa x_2 &= \frac{1}{\psi'} \bar{x}_2 + \frac{2\theta}{\psi'^2} \bar{x}_3 + \frac{1}{\psi'^3} \left(\frac{6}{5} P_2 + \frac{3}{2} \theta^2 \right) \bar{x}_4, \\
 (32) \quad \kappa x_3 &= \frac{1}{\psi'^2} \bar{x}_3 + \frac{1}{\psi'^3} \frac{3\theta}{2} \bar{x}_4, \\
 \kappa x_4 &= \frac{1}{\psi'^3} \bar{x}_4.
 \end{aligned}$$

¹ Proj. Diff. Geom., pp. 239-241. The notation is changed slightly to avoid conflict.

The inverse transformation is

$$\begin{aligned}
 \kappa \bar{x}_1 &= \frac{1}{\psi'^3} \left[x_1 - \frac{3\theta}{2} x_2 - \left(\frac{4}{5} P_2 - \frac{3}{2} \theta^2 \right) x_3 - \frac{3}{20} (2P_2' - 12\theta P_2 + 5\theta^3) x_4 \right], \\
 (33) \quad \kappa \bar{x}_2 &= \frac{1}{\psi'^2} \left[\begin{array}{cc} x_2 - 2\theta x_3 & - \left(\frac{6}{5} P_2 - \frac{3}{2} \theta^2 \right) x_4 \end{array} \right], \\
 \kappa \bar{x}_3 &= \frac{1}{\psi'} \left[\begin{array}{cc} x_3 & - \frac{3}{2} \theta x_4 \end{array} \right], \\
 \kappa \bar{x}_4 &= x_4.
 \end{aligned}$$

In these transformations ψ' and θ are constants depending upon the transformation of the independent parameter s necessary to make $P_2 = 0$, and upon the value of s at the particular point P , the common vertex of the two tetrahedra.

3. FUNDAMENTAL EXPANSIONS FOR THE COORDINATES AND DIRECTION COSINES AT A POINT ON THE GIVEN CURVE IN THE NEIGHBORHOOD OF P .

Since our study will now be concerned with differential properties of the given curve in the neighborhood of P , we shall have only occasional use for the general fixed system of coordinates. It will therefore be convenient to change the significance of $\alpha, \beta, \gamma, l, m, n, \lambda, \mu, \nu$, so that they shall be direction cosines referred to the local trihedral system, the $\xi\eta\zeta$ -system. We shall henceforward measure s from P .

We shall have occasion to use the expansions for the coordinates ξ, η, ζ of a point on the curve in the neighborhood of P to terms of the fifth order, and we shall also need to know the expansions for α, β, γ to terms of the fourth order. We proceed to obtain these expansions. The developments will be of the general form

$$(34) \quad \xi = \xi_0 + \xi'_1 s + \frac{\xi''_2}{2!} s^2 + \frac{\xi'''_3}{3!} s^3 + \frac{\xi^{iv}_4}{4!} s^4 + \frac{\xi^v_5}{5!} s^5 + \dots$$

If in (1) we replace x, y, z , by ξ, η, ζ , and extend the formulas to terms involving the fifth derivative, and then specialize α, β, γ , etc., to refer to the local trihedral, so that at P ,

$$(35) \quad \alpha = m = \nu = 1, \quad \beta = \gamma = l = n = \lambda = \mu = 0, \quad \xi = \eta = \zeta = 0,$$

we obtain the desired expansions by substitution in the equations of the form (34). After the introduction of the notation,

$$\begin{aligned}
k_1 &= \frac{1}{24r^2\tau^2} (2r'\tau + r\tau'), \\
k_2 &= \frac{1}{24r^3\tau^2} (r^2 + \tau^2 + rr''\tau^2 - 2r'^2\tau^2), \\
(36) \quad k_3 &= \frac{1}{24r^4\tau^2} (r^2 + \tau^2 + 4rr''\tau^2 - 11r'^2\tau^2), \\
k_4 &= \frac{1}{24r^4\tau^3} (3r^3\tau' + 3r^2r'\tau - r^2r''\tau^3 + 6rr'r''\tau^3 + 6r'\tau^3 - 6r'^3\tau^3), \\
k_5 &= \frac{1}{24r^3\tau^3} (r^2 + \tau^2 + r^2\tau r'' - 2r^2\tau'^2 - 3rr'\tau\tau' + 3rr''\tau^2 - 6r'^2\tau^2),
\end{aligned}$$

we have

$$\begin{aligned}
(37) \quad \xi &= s - \frac{1}{6r^2}s^3 + \frac{r'}{8r^3}s^4 + \frac{k_3}{5}s^5 + \dots, \\
\eta &= \frac{1}{2r}s^2 - \frac{r'}{6r^2}s^3 - k_2s^4 + \frac{k_4}{5}s^5 + \dots, \\
\zeta &= -\frac{1}{6r\tau}s^3 + k_1s^4 + \frac{k_5}{5}s^5 + \dots,
\end{aligned}$$

and

$$\begin{aligned}
(38) \quad \alpha &= 1 - \frac{1}{2r^2}s^2 + \frac{r'}{2r^3}s^3 + k_3s^4 + \dots, \\
\beta &= \frac{1}{r}s - \frac{r'}{2r^2}s^2 - 4k_2s^3 + k_4s^4 + \dots, \\
\gamma &= -\frac{1}{2r\tau}s^2 + 4k_1s^3 + k_5s^4 + \dots.
\end{aligned}$$

Expansions for $l, m, n, \lambda, \mu, \nu$ will be needed only up to terms in s^2 . These are easily found to be

$$\begin{aligned}
(39) \quad l &= -\frac{1}{r}s + \frac{r'}{2r^2}s^2 + \dots, \\
m &= 1 - \frac{r^2 + \tau^2}{2r^2\tau^2}s^2 + \dots, \\
n &= -\frac{1}{\tau}s + \frac{\tau'}{2\tau^2}s^2 + \dots,
\end{aligned}$$

and

$$\begin{aligned}
(40) \quad \lambda &= -\frac{1}{2r\tau}s^2 + \dots, \\
\mu &= \frac{1}{\tau}s - \frac{\tau'}{2\tau^2}s^2 + \dots, \\
\nu &= 1 - \frac{1}{2\tau^2}s^2 + \dots.
\end{aligned}$$

4. THE OSCULATING RULED QUADRIC OF THE SURFACE OF BINORMALS.

The binormals of the given curve form a ruled surface of which the curve itself is the line of striction.¹ Its parametric equations are

$$(41) \quad \begin{aligned} X &= x + \lambda t, \\ Y &= y + \mu t, \\ Z &= z + \nu t, \end{aligned}$$

where X, Y, Z , are the coordinates of a point on the surface; x, y, z , those of a point on the curve; and λ, μ, ν are the direction cosines of the binormal to the curve at the point (x, y, z) , all referred to the same fixed system; and t is the directed distance from (x, y, z) to the point (X, Y, Z) of the surface.

If we take the local trihedral as the fixed system and consider points on binormals in the neighborhood of P , we shall have by (37) and (40), up to terms in s^2 ,

$$(42) \quad \begin{aligned} \xi &= s - \frac{1}{2r\tau} s^2 t + \dots, \\ \eta &= \frac{1}{2r} s^2 + \frac{1}{\tau} s t - \frac{\tau'}{2\tau^2} s^2 t + \dots, \\ \zeta &= t - \frac{1}{2\tau^2} s^2 t + \dots. \end{aligned}$$

We seek to determine the ruled quadric which osculates the surface of binormals along the binormal at P . This amounts to finding the ruled quadric which is determined by three binormals in the neighborhood of P and taking the limiting position as s approaches zero. Or, what amounts to the same thing, we must find the quadric surface whose equation is satisfied identically by (42) up to terms in s^2 for all values of t .

Since for $s = 0$ every point on the ζ -axis is on the surface, the equation is of the form

$$(43) \quad A_{11}\xi^2 + A_{22}\eta^2 + 2A_{12}\xi\eta + 2A_{13}\xi\zeta + 2A_{23}\eta\zeta + 2A_{10}\xi + 2A_{20}\eta = 0.$$

Substituting (42) in (43) and equating to zero the coefficients of s and s^2 , we obtain the equations of condition,

$$(44) \quad \begin{aligned} A_{11} + \frac{A_{22}}{\tau^2} t^2 + \frac{2A_{12}}{\tau} t - \frac{A_{13}}{r\tau} t^2 + \frac{A_{23}}{r} t - \frac{A_{10}}{r\tau} t + \frac{A_{20}}{r} - \frac{A_{20}\tau'}{\tau^2} t &= 0, \\ 2A_{13}t + \frac{2A_{23}}{\tau} t^2 + 2A_{10} + \frac{2A_{20}}{\tau} t &= 0. \end{aligned}$$

Since these must be satisfied for all values of t , we must have

$$(45) \quad A_{11} : A_{22} : A_{12} : A_{13} : A_{23} : A_{10} : A_{20} = 2\tau : 2\tau : -r\tau' : 2r : 0 : 0 : -2r\tau.$$

¹ Schell, Allgemeine Theorie der Curven doppelter Krümmung (1898), p. 116.

The equation of the osculating ruled quadric of the surface of binormals is therefore

$$(46) \quad \tau(\xi^2 + \eta^2) - r\tau'\xi\eta + 2r\xi\zeta - 2r\tau\eta = 0.$$

The discriminating cubic is

$$(47) \quad \varphi^3 - 2\tau\varphi^2 + \left(\tau^2 - r^2 - \frac{r^2\tau'^2}{4}\right)\varphi + r^2\tau = 0;$$

it has evidently only two roots of the same sign whatever be the sign of τ . The surface is therefore an hyperboloid of one sheet. Its center is the point $(0, r, r\tau'/2)$, which is at the distance $(r/2)\sqrt{\tau'^2 + 4}$ from P .

The ruled quadric which osculates the surface of binormals along the binormal at P is an hyperboloid of one sheet with its center upon the axis of curvature of the given curve corresponding to the point P ; the diameter of the quadric which passes through P lies in the normal plane and has its slope with respect to the principal normal equal to $\tau'/2$. At a point of the curve at which τ' is zero, the center of the quadric coincides with the center of curvature of the given curve.

The tangent plane to the quadric (46) at the point P is the rectifying plane, $\eta = 0$. This plane cuts the surface in the binormal and the line

$$(48) \quad \eta = 0, \quad \tau\xi + 2r\zeta = 0.$$

This line, the second generator through P , has a slope with respect to the tangent equal to $-\tau/2r$, just exactly half that of the rectifying line,¹ the generator of the rectifying developable which lies in the same rectifying plane.

If the equation of the quadric (46) be written referred to its own axes, it is found that the product of its semiaxes is equal numerically to the product of the radius of torsion by the square of the radius of curvature, $|r^2\tau|$.

If in (46) we replace the coefficient of $\xi\eta$, which involves τ' , by a parameter 2ϵ , we shall have the equation of a one-parameter family of ruled quadrics tangent to the surface of binormals along the binormal through P , including the osculating quadric for $\epsilon = -r\tau'/2$. The members of this family are hyperboloids of one sheet with centers $(0, r, -\epsilon)$.

Hence, the axis of curvature corresponding to the point P of a given curve is the locus of the centers of the one-parameter family of hyperboloids of one sheet which are tangent to the surface of binormals of the given curve along the binormal through P . The polar developable of the given curve is the locus of the centers as the point P moves along the curve.

¹ Lilienthal, Vorlesungen über Differentialgeometrie, I, p. 213.

5. THE OSCULATING RULED QUADRIC OF THE SURFACE OF PRINCIPAL NORMALS.

The parametric equations of the ruled surface of principal normals referred to the fixed system of coordinates are

$$(49) \quad \begin{aligned} X &= x + lt, \\ Y &= y + mt, \\ Z &= z + nt. \end{aligned}$$

Referred to the local trihedral and expressed in powers of s , we have by (37) and (39), in the neighborhood of P ,

$$(50) \quad \begin{aligned} \xi &= s - \frac{1}{r}st + \frac{r'}{2r^2}s^2t + \dots, \\ \eta &= \frac{1}{2r}s^2 + t - \frac{r^2 + \tau^2}{2r^2\tau^2}s^2t + \dots, \\ \zeta &= -\frac{1}{\tau}st + \frac{\tau'}{2\tau^2}s^2t + \dots. \end{aligned}$$

Proceeding exactly as in the preceding section, we find for the equation of the osculating ruled quadric,

$$(51) \quad r'\tau^2\zeta^2 - 2r^2\xi\eta - r^2\tau'\xi\zeta + 2r\tau\eta\zeta - 2r^2\tau\zeta = 0.$$

The discriminating cubic is

$$(52) \quad \varphi^3 - \frac{r'\tau}{r^2}\varphi^2 - \frac{1}{4r^2\tau^2}(4r^2 + 4\tau^2 + r^2\tau'^2)\varphi - \frac{r\tau' - r'\tau}{r^2\tau^2} = 0.$$

Unless $r\tau' - r'\tau = 0$, the discriminating cubic has two roots of one sign and one of the other. Hence, the ruled quadric which osculates the surface of principal normals along the principal normal through P is in general an hyperboloid of one sheet; it is an hyperbolic paraboloid at points of the curve for which $r\tau' - r'\tau = 0$, and an equilateral hyperbolic paraboloid at points where $r' = \tau' = 0$.

The center of the quadric is the point

$$\left(\frac{r\tau}{r'\tau - r\tau'}, -\frac{r^2\tau'}{2(r'\tau - r\tau')}, \frac{r^2}{r'\tau - r\tau'} \right),$$

and the ratios of the direction cosines of the diameter through P are

$$2\tau : -r\tau' : 2r.$$

Comparison with the results of the preceding section shows that the projection upon the normal plane of the diameter through P of the osculating ruled quadric of the surface of principal normals is perpendicular to the diameter through P of the ruled quadric which osculates the surface of binormals; and that the projection of the diameter through P upon the rectifying plane is perpendicular to the rectifying line.

The tangent plane to the surface (51) at P is the osculating plane, $\zeta = 0$. This cuts the surface in

$$(53) \quad \zeta = 0, \quad \xi\eta = 0.$$

The generators through P are therefore the principal normal, of course, and the tangent to the curve at P . A purely geometrical argument can be set up for this statement, derived so simply analytically.

6. THE PROJECTION UPON THE OSCULATING PLANE; ITS OSCULATING CONIC.

The coordinates of a point of the orthogonal projection of the given curve upon its osculating plane at P are given by the first two of equations (37). The osculating conic of this plane projection must have five points in common with the projection at P ; its equation must therefore be satisfied identically by (37) up to terms in s^4 . These conditions are sufficient to determine the coefficients in the equation, which is found to be

$$(54) \quad 9\xi^2 + \left(9 + 2r'^2 - 3rr'' - \frac{3r^2}{\tau^2}\right)\eta^2 - 6r'\xi\eta - 18r\eta = 0.$$

The discriminant of this equation may be written

$$(55) \quad 9\left(w - \frac{3r^2}{\tau^2}\right),$$

where

$$(56) \quad w = 9 + r'^2 - 3rr''$$

is independent of the torsion.

The conic is an ellipse, parabola, or hyperbola according as

$$(57) \quad w \begin{cases} \geq \\ \equiv \\ \leq \end{cases} \frac{3r^2}{\tau^2}.$$

Its center is the point

$$\left(\frac{3rr'}{w - \frac{3r^2}{\tau^2}}, \frac{9r}{w - \frac{3r^2}{\tau^2}}, 0 \right),$$

and the diameter through P is the line

$$(58) \quad 3\xi - r'\eta = 0, \quad \zeta = 0.$$

This is the axis of aberrancy (*axis de déviation*) of Transon.¹ It is to be noted that the r' in (58) is that belonging to the space curve at P , not to its plane projection. Transon's result shows that the axis of aberrancy has an equation of the same form, where r' refers to the plane curve. In general then, the axis of aberrancy of the projection of a given space curve upon its osculating plane is independent of the torsion, and is the

¹ Transon, *Journal de Mathématiques*, VI (1841), p. 193.

Wilczynski, *Bulletin of the American Mathematical Society*, XXII, No. 7 (1916), p. 321.

same as that of any plane curve in the osculating plane having at the point P the same tangent as the space curve and the same value of r' , where r' in the one case is the derivative of the radius of curvature of the space curve with respect to the arc length of that curve, and in the other is the derivative of the radius of curvature of the plane curve with respect to its arc length.

7. THE OSCULATING LINEAR COMPLEX.

By the osculating linear complex of a curve is meant the linear complex determined by five consecutive tangents to the curve. Referred to the fixed system, the parametric equations of the tangent surface are

$$(59) \quad \begin{aligned} X &= x + \alpha t, \\ Y &= y + \beta t, \\ Z &= z + \gamma t. \end{aligned}$$

Referred to the local trihedral and expressed in powers of s , we have by (37) and (38), in the neighborhood of P ,

$$(60) \quad \begin{aligned} \xi &= s - \frac{1}{6r^2}s^3 + \frac{r'}{8r^3}s^4 + \dots + \left(1 - \frac{1}{2r^2}s^2 + \frac{r'}{2r^3}s^3 + k_3s^4 + \dots\right)t, \\ \eta &= \frac{1}{2r}s^2 - \frac{r'}{6r^2}s^3 - k_2s^4 + \dots + \left(\frac{1}{r}s - \frac{r'}{2r^2}s^2 - 4k_2s^3 + k_4s^4 + \dots\right)t, \\ \zeta &= -\frac{1}{6r\tau}s^3 + k_1s^4 + \dots + \left(-\frac{1}{2r\tau}s^2 + 4k_1s^3 + k_5s^4 + \dots\right)t. \end{aligned}$$

If in (23) we take for ξ, η, ζ the values obtained from (60) for $t = 0$, and for ξ', η', ζ' the corresponding values for $t = 1$, we find as the line coordinates of tangents to the curve in the neighborhood of P the following values,

$$(61) \quad \begin{aligned} \kappa\pi_{12} &= \frac{1}{2r}s^2 - \frac{r'}{3r^2}s^3 + \left(\frac{1}{12r^3} - 3k_2\right)s^4 + \dots, \\ \kappa\pi_{13} &= -\frac{1}{3r\tau}s^3 + 3k_1s^4 + \dots, \\ \kappa\pi_{14} &= -1 + \frac{1}{2r^2}s^2 - \frac{r'}{2r^3}s^3 - k_3s^4 + \dots, \\ \kappa\pi_{23} &= -\frac{1}{12r^3\tau}s^4 + \dots, \\ \kappa\pi_{42} &= \frac{1}{r}s - \frac{r'}{2r^2}s^2 - 4k_2s^3 + k_4s^4 + \dots, \\ \kappa\pi_{34} &= \frac{1}{2r\tau}s^2 - 4k_1s^3 - k_5s^4 + \dots. \end{aligned}$$

The linear homogeneous equation between $\pi_{12}, \dots, \pi_{34}$ which is satisfied by the expressions (61) up to terms of the fourth order in s determines the osculating linear complex at P . Instead of determining this equation directly, we shall obtain it by transformation from the equation referred to the local tetrahedron $P_x P_y P_z P_s$. Wilczynski has found the latter equation to be¹

$$(62) \quad \omega_{14} - \omega_{23} - 2P_2\omega_{34} = 0.$$

Applying the transformation (29), the equation referred to the local trihedral is found to have the form

$$(63) \quad 4\tau\pi_{12} + 2\tau\tau'\pi_{13} - r(4 + \tau'^2 - 2\tau\tau'')\pi_{23} - 4\tau^2\pi_{34} = 0.$$

We may verify by direct substitution of (61) in (63) that this equation is satisfied up to terms of the fourth order. For convenience we introduce the notation

$$(64) \quad \begin{aligned} 2u &= \frac{r}{\tau}(4 + \tau'^2 - 2\tau\tau''), \\ K &= 4 + \tau'^2 + u^2, \end{aligned}$$

and write (63) in the form

$$(65) \quad 2\pi_{12} + \tau'\pi_{13} - u\pi_{23} - 2\tau\pi_{34} = 0.$$

The equations of the axis of this complex are²

$$(66) \quad \frac{\xi + \frac{2\tau\tau'}{K}}{u} = \frac{\eta - \frac{2\tau u}{K}}{\tau'} = \frac{\zeta}{-2},$$

and the parameter of the complex is given by³

$$(67) \quad k = -\frac{4\tau}{K}.$$

If the complex is special, that is, if all of its lines pass through a common axis, we must have $k = 0$.⁴ The same condition may be expressed in the form $P_2 = \infty$.⁵

From either form of the conditions it follows that at a point where r and τ are finite and not zero, and τ' and τ'' are finite, the osculating linear complex cannot be special.

For a left-handed coordinate system, the complex is right handed or left handed according as k is positive or negative,⁶ or according as τ is negative or positive. This agrees with the fact that for a fixed coordinate system and a local trihedral system both

¹ Proj. Diff. Geom., p. 253.

² Plücker, *Neue Geometrie des Raumes*, I, pp. 26–32.

³ Plücker, loc. cit., p. 41.

⁴ Plücker, loc. cit., p. 57.

⁵ Proj. Diff. Geom., p. 253.

⁶ Plücker, loc. cit., p. 57.

of which are left-handed the curve itself is right-handed or left-handed according as τ is negative or positive.¹ For right-handed coordinate systems the signs of k and τ are reversed.

The null-plane of a point (ξ', η', ζ') with respect to the complex (65) has the equation

$$(68) \quad 2(\xi\eta' - \xi'\eta) + \tau'(\xi\zeta' - \xi'\zeta) - u(\eta\zeta' - \eta'\zeta) - 2\tau(\zeta - \zeta') = 0.$$

This shows, as is evident geometrically, that the osculating plane is the null-plane of the point P .

Imposing the condition that (68) shall represent the normal plane, we find for its null-point the point $(0, 2\tau/u, 0)$ on the principal normal; similarly the null-point of the rectifying plane is the point $(-2\tau/\tau', 0, 0)$ on the tangent; so that the null-points of the local trihedral planes with respect to the osculating linear complex all lie in the osculating plane.

The equation (65) is the equation of a one-parameter family of penosculating complexes, having contact of the third order with the curve at P , if u be regarded as an arbitrary parameter. Equation (66) now represents a one-parameter family of lines, forming the ruled surface of axes of the one-parameter family of linear complexes. This surface is a cylindroid,² or cubical conoid.

The central plane³ of the cylindroid, to which all its generators are parallel, is

$$(69) \quad 2\eta + \tau'\zeta = 0.$$

The height of the cylindroid, the distance between the bounding planes parallel to (69), is

$$(70) \quad h = \frac{4\tau}{\sqrt{4 + \tau'^2}}.$$

The two generators lying in the central plane are,

$$(71) \quad \begin{aligned} \text{for } u = 0, \quad \xi + \frac{2\tau\tau'}{4 + \tau'^2} = 0, \quad 2\eta + \tau'\zeta = 0, \\ \text{for } u = \infty, \quad \eta = 0, \quad \zeta = 0. \end{aligned}$$

The axis of the cylindroid must pass through the intersection of these two generators and be perpendicular to the central plane; its equations are therefore

$$(72) \quad \xi + \frac{2\tau\tau'}{4 + \tau'^2} = 0, \quad \tau'\eta - 2\zeta = 0.$$

Plücker calls this line the axis of the congruence determined by the family of com-

¹ Eisenhart, *Differential Geometry*, p. 19.

² Plücker, *loc. cit.*, pp. 95-97.

³ Plücker, *loc. cit.*, p. 63.

plexes.¹ This axis is parallel to the diameter through P of the ruled quadric which osculates the surface of binormals, as is seen by reference to section four.

The directrices of the congruence are the axes of the two special complexes of the family.² By (67) we see that the condition that the complex be special is $K = \infty$, therefore $u = \pm \infty$. In this case we see, by (71), that the corresponding axis is the tangent to the curve at P . The congruence therefore has coincident directrices and is special. When the directrices coincide, the line in which they coincide is itself a line of the congruence and is therefore common to all the complexes.³ This agrees precisely with the fact that the tangent is a line of each of the penosculating complexes.

8. THE OSCULATING QUADRIC CONE.

The term osculating cone has been used to refer to several cones differently related to the space curve. Saint-Venant⁴ has used it to refer to the oblique circular cone having the corresponding tangent and rectifying line at P as element and axis respectively, the circular base being perpendicular to the tangent; or to the cone of revolution having the same element and axis. In each of these cases the cone is thought of as osculating the tangent surface, but the cone of revolution so defined is the same as one determined as having its vertex on the curve and three generators parallel to three successive tangents to the curve.⁵

The same name has been used to refer to the cone of revolution having the osculating circle for a base and having third order contact with the curve.⁶ The cone of revolution having its vertex on the curve and having third order contact with the curve at that point has also been called the osculating cone.⁷

We shall follow Wilczynski in defining the osculating cone as the quadric cone with vertex on the curve having the closest possible contact with the curve.⁸

If we write the equation of a quadric cone having its vertex at P in the most general form and substitute equations (37), we shall find that there is a family of cones having fifth order contact with the curve at P and among these one having contact of the sixth order. This is the osculating cone.

Instead of determining the equation of this cone directly, we shall get it by transformation from Wilczynski's result in the projective case. He finds the equation in the form⁹

$$(73) \quad 3\bar{x}_2\bar{x}_4 - 2\bar{x}_3^2 = 0,$$

¹ Plücker, loc. cit., p. 66.

² Jessop, Treatise on the Line Complex, p. 60.

³ Plücker, loc. cit., p. 74.

⁴ Saint-Venant, Journal de l'école polytechnique, Cah. 30 (1845), pp. 40-43.

⁵ Scheffers, Anwendung der Differential und Integral-Rechnung auf Geometrie, I, pp. 255-257.

⁶ Schell, Allgemeine Theorie der Curven doppelter Krümmung, pp. 67-68.

⁷ Scheffers, loc. cit., p. 260.

⁸ Proj. Diff. Geom., p. 250.

⁹ Proj. Diff. Geom., p. 249.

referred to a specialized tetrahedron where the independent parameter has been so chosen that $P_2 = 0$. If this specialization be removed, the equation referred to the general local tetrahedron is found by applying the transformation (33) to (73); it is

$$(74) \quad 3x_2x_4 - 2x_3^2 - \frac{18}{5}P_2x_4^2 = 0.$$

Applying the transformation (22), we find as the equation of the osculating cone referred to the local trihedral,

$$(75) \quad 80r^2\eta^2 + 120r\tau\xi\zeta - 20r(2r'\tau - 3r\tau')\eta\zeta - J\zeta^2 = 0,$$

where

$$(76) \quad J = 36\tau^2 - 84r^2 + 4r'^2\tau^2 + 12rr'\tau\tau' - 27r^2\tau'^2 - 12rr''\tau^2 + 36r^2\tau\tau''.$$

If (37) be substituted in (75) we shall see that it is satisfied identically up to terms in s^6 ; the contact is therefore of the sixth order.

The osculating plane is tangent to the cone along the tangent to the curve at P . Sections of the cone parallel to the osculating plane are parabolas whose vertices lie upon the generator in the normal plane,

$$(77) \quad \xi = 0, \quad 8r\eta - (2r'\tau - 3r\tau')\zeta = 0.$$

The normal plane intersects the cone in two generators which are real and distinct, coincident, or imaginary according as

$$5(2r'\tau - 3r\tau')^2 + 4J \gtrless 0;$$

under the same conditions sections parallel to the normal plane are hyperbolas, parabolas, or ellipses, the locus of the centers of these sections being the line

$$(78) \quad \frac{\xi}{5(2r'\tau - 3r\tau')^2 + 4J} = \frac{\eta}{30\tau(2r'\tau - 3r\tau')} = \frac{\zeta}{240r\tau},$$

the projection of which upon the normal plane is the locus (77).

The rectifying plane cuts the cone in two real generators, one of which is the tangent and the other the line

$$(79) \quad \eta = 0, \quad 120r\tau\xi - J\zeta = 0,$$

while sections parallel to this plane are hyperbolas whose centers lie in the osculating plane, upon the line

$$(80) \quad 6\tau\xi - (2r'\tau - 3r\tau')\eta = 0, \quad \zeta = 0.$$

9. THE OSCULATING TWISTED CUBIC.

Since a twisted cubic is the algebraic space curve of lowest order, the osculating twisted cubic of a space curve may be regarded as replacing the osculating conic of a plane curve. A twisted cubic may be defined geometrically as a part of the complete

intersection of two ruled quadrics which have a generator in common. In particular one of these quadrics may be a cone, whence the term cubical conic sometimes applied to the curve.¹

The homogeneous coordinates of a point on the curve may be expressed as cubic polynomials of a single parameter.² A twisted cubic is determined by six points, no four of which are coplanar.³ The secants of the curve which pass through a common point on the curve are the generators of a unique quadric cone.⁴

The osculating twisted cubic of a given curve will have fifth order contact at P ; since it lies upon a unique quadric cone with vertex at P , and since the osculating cone has sixth order contact at P , the osculating twisted cubic lies upon the osculating cone.

We might determine a second ruled quadric having at least fifth order contact with the given curve at P and having a generator in common with the osculating cone (75), and thence determine the osculating cubic as the remaining portion of the intersection of the two surfaces. It is simpler however to derive these surfaces and the cubic from the corresponding formulae of projective geometry referred to the local tetrahedron, as developed by Wilczynski.⁵ He finds for the parametric equations of the osculating twisted cubic, referred to $P_Y P_Z P_\rho P_\sigma$,

$$\begin{aligned} \kappa x_1 &= 12(P_3 - P_2') + 24P_2 t + 15t^3, \\ \kappa x_2 &= 24P_2 + 30t^2, \\ \kappa x_3 &= 30t, \\ \kappa x_4 &= 20. \end{aligned} \tag{81}$$

Applying the transformation (22), we obtain the equations referred to the local trihedral,

$$\begin{aligned} \xi &= \frac{24P_2 + 20\left(3p_2 - \frac{1}{r^2}\right) + 60p_1 t + 30t^2}{12(P_3 - P_2') + 20p_3 + 24p_1 P_2 + (24P_2 + 30p_2)t + 30p_1 t^2 + 15t^3}, \\ \eta &= \frac{\frac{10}{r\tau}(2p_1\tau + \tau' + 3\tau t)}{\text{same denominator}}, \\ \zeta &= \frac{-\frac{20}{r\tau}}{\text{same denominator}}. \end{aligned} \tag{82}$$

¹ Staude, *Analytische Geometrie der kubischen Kegelschnitte*, p. 1.

² Staude, *loc. cit.*, p. 2.

³ Chasles, *Journal de Mathématiques*, 2d serie, II (1857), p. 397.

⁴ Seydewitz, *Archiv für Math. u. Phys.*, 10 (1847), p. 207.

⁵ *Proj. Diff. Geom.*, p. 250.

Twisted cubics are classified according to the nature of their points at infinity.¹ The cubical ellipse has one real and two imaginary points at infinity; the cubical hyperbola has three real and distinct points; the cubical hyperbolic parabola has three real points, two of which are coincident; and the cubical parabola has three coincident points, at infinity.

The points at infinity in (82) are given by values of t for which the denominator is equal to zero; that is, by the cubic equation

$$(83) \quad 12(P_3 - P_2') + 20p_3 + 24p_1P_2 + (24P_2 + 30p_2)t + 30p_1t^2 + 15t^3 = 0.$$

If we denote by H and G respectively the seminvariants of degree two and three of (83), we have

$$(84) \quad \begin{aligned} H &= 10(12P_2 + 15p_2 - 10p_1^2), \\ G &= 100[27(P_3 - P_2') + 18P_2p_1 + 20p_1^3 - 45p_1p_2 + 45p_3]. \end{aligned}$$

The roots of (83) may then be classified as follows:²

$$(85) \quad \begin{aligned} &< 0, \text{ roots real and distinct,} \\ G^2 + 4H^3 &= 0, \text{ two roots equal; three equal if } G = H = 0, \\ &> 0, \text{ one root real, two imaginary.} \end{aligned}$$

The relations (85) enable us to classify the osculating cubic (82).

10. THE OSCULATING CONIC OF THE SPACE CURVE.

The tangent surface of a space cubic curve is a surface of the fourth order. An osculating plane intersects it in the tangent counted twice and a conic.³ We may also obtain the same conic as the envelope of the lines in which a moving osculating plane intersects a fixed osculating plane. Staude calls each of these lines an axis and the conic the axis-conic belonging to the fixed osculating plane.⁴

Consider a given space curve. Determine its osculating twisted cubic at P and the axis-conic of the latter which lies in the osculating plane of P . Wilczynski calls this conic the osculating conic⁵ of the given curve at P , and he finds as its equations, referred to the local tetrahedron,

$$(86) \quad 40x_1x_3 - 15x_2^2 - 32P_2x_3^2 = 0, \quad x_4 = 0.$$

Making use of the transformation (22), we find that the osculating conic, referred to the local trihedron, is given by

$$(87) \quad 15\xi^2 - 20p_1r\xi\eta + r^2(32P_2 - 20p_1^2 + 40p_2)\eta^2 - 40r\eta = 0, \quad \zeta = 0.$$

¹ Seydewitz, loc. cit., p. 211.

² Burnside and Panton, Theory of Equations, I, p. 84.

³ Möbius, Werke, I, p. 121.

⁴ Staude, loc. cit., p. 46.

⁵ Proj. Diff. Geom., p. 250.

The discriminant of this conic is equal to $-6000r^2$; the conic is therefore not degenerate at any regular point of the curve. The discriminant of the quadratic terms is $4r^2H$, where H has the significance of (84). The conic is therefore an ellipse, parabola, or hyperbola according as $H \gtrless 0$.

The center of this conic is the point $(100p_1/H, 150/rH, 0)$, and the diameter through P is the line

$$(88) \quad 3\xi - 2p_1r\eta = 0, \quad \zeta = 0.$$

The osculating conic of the space curve as here defined is to be carefully distinguished from the osculating conic of the projection of the space curve upon its osculating plane which was studied in a preceding section. The diameter through P for the latter conic, which is the axis of aberrancy of the plane projection, depends upon r' only, as shown by (58); for the former conic it depends also upon r , τ , and τ' , as (88) shows.

When $H \neq 0$, the osculating conic is central. If we denote by A and B the coefficients of the squared terms when the conic is referred to its own axes as coordinate axes, we have

$$A + B = 15 + r^2(32P_2 - 20p_1^2 + 40p_2),$$

$$AB = 4r^2H.$$

The osculating conic is therefore an equilateral hyperbola if

$$15 + r^2(32P_2 - 20p_1^2 + 40p_2) = 0,$$

and a circle if

$$r^4(32P_2 - 20p_1^2 + 40p_2)^2 - 40r^2(24P_2 - 25p_1^2 + 30p_2) + 225 = 0.$$

11. THE RELATION BETWEEN THE OSCULATING TWISTED CUBIC AND THE OSCULATING CONIC.

The relations between the osculating twisted cubic and the osculating conic for different values of H and G , (84), can be tabulated thus:

$G^2 + 4H^3$	H	G	Osculating cubic	Osculating conic
> 0	> 0	$\gtrless 0$	Cubical ellipse	Ellipse
> 0	$= 0$	$\neq 0$	Cubical ellipse	Parabola
> 0	< 0	> 0	Cubical ellipse	Hyperbola
$= 0$	< 0	$\neq 0$	Cubical hyperbolic parabola	Hyperbola
$= 0$	$= 0$	$\neq 0$	Cubical parabola	Parabola
< 0	< 0	$\gtrless 0$	Cubical hyperbola	Hyperbola

These results are in agreement with the general relations between the species of twisted cubics and of their osculating conics found by Cremona.¹

¹ Cremona, *Annali di Matematica*, I ser., 2 (1859), pp. 201-7. *Opere Matematiche*, I, pp. 100-7.

12. OSCULANTS CONSIDERED AS IMAGES OF r AND τ AND THEIR DERIVATIVES.

We have seen that certain osculants of a space curve are determined if the radii of curvature and of torsion and certain of their derivatives at the point of contact are known. We shall say that an osculant serves as an image of a given intrinsic function if the osculant depends upon this function and upon none of higher order, though it may depend upon other intrinsic functions of the same or lower order associated with osculants already discussed.

Thus, the osculating circle is the osculant which furnishes an image for r , its radius being r . Saint-Venant has given a geometrical image for τ in connection with the cone of revolution osculating the tangent surface.¹ The osculating circular helix also serves as a convenient image of τ since its axis and its pitch are determined by r and τ .² The osculating sphere may be considered as the image of r' since its radius is given in terms of r , τ , and r' . The cone of revolution having its vertex on the axis of curvature and having third order contact with the given curve may serve equally well as the image of r' , since the distance of its vertex from the osculating plane is $|r^2/r'\tau|$.³

This seems to be about as far as the attempt to secure geometrical images of derivatives of the intrinsic functions r and τ has gone. Our results enable us to make an extension.

We found, in (46), that the osculating ruled quadric of the surface of binormals depends upon r , τ , and τ' only. We may therefore regard this osculant as an image of τ' . If we prefer, we may consider the image of τ' as furnished by the diameter of this quadric which passes through P , since it has been shown to lie in the normal plane and to have its slope with respect to the principal normal equal to $\tau'/2$.

The osculating conic (54) of the projection of the given curve upon its osculating plane depends upon r , τ , r' , and r'' . The axis of aberrancy (58) depends upon r' only and may be taken as another image of r' ; the conic itself, or its center, may be considered as an image of r'' .

The osculating linear complex may be thought of as an image of τ'' , since it depends only upon r , τ , τ' , and τ'' , as we saw in (63). Or we may consider the axis of the complex, as given by (66), as the image of τ'' , since it is determined by the same intrinsic functions which determine the complex. Still another image of τ'' is furnished by the osculating quadric cone (75), which depends upon r , τ , r' , τ' , r'' , and τ'' . Or the quadric cone may be considered as an image of r'' if τ'' is understood already to have found its image in the osculating linear complex.

If we notice that, in equations (82), τ''' enters only through P_3 and P_2' and is eliminated from the combination $P_3 - P_2'$, we see that the osculating twisted cubic depends upon r , τ , r' , τ' , r'' , τ'' , and r''' . It may be regarded as an image of r''' . If in (82) we replace r''' by an arbitrary parameter, we obtain the equations of a one-parameter family

¹ Saint-Venant, loc. cit., p. 54; Lilienthal, loc. cit., p. 294.

² Scheffers, loc. cit., I, p. 197.

³ Schell, loc. cit., pp. 67-68.

of penosculating twisted cubics having fourth order contact with the curve; they all lie upon the same osculating cone and all belong to the same osculating linear complex.

These relations between the intrinsic functions and the osculants may be expressed in a somewhat different way. Consider the family of all curves through P having at that point the same local trihedral. These curves may be subdivided into classes in accordance with the following scheme.

Curves having at P the same	will have the same
r	Osculating circle.
r, τ	Cone of revolution osculating the tangent surface along the tangent at P . ¹ Cone of revolution with vertex at P and having third order contact with the curve. ²
r, τ, r'	Osculating circular helix having same τ as the curve. ³
r, τ, τ'	Osculating sphere.
r, τ, r', τ'	Ruled quadric osculating the surface of binormals (46).
r, τ, r', r''	Ruled quadric osculating the surface of principal normals (51).
r, τ, r', r''	Osculating conic of the projection upon the osculating plane (54).
r, τ, τ', τ''	Osculating linear complex (63).
$r, \tau, r', \tau', r'', \tau''$	Osculating quadric cone (75).
$r, \tau, r', \tau', r'', \tau''$	Osculating conic of the space curve (87).
$r, \tau, r', \tau', r'', \tau'', r'''$	Osculating twisted cubic (82).

13. APPLICATION OF RESULTS TO SOME SIMPLE SPECIAL CURVES.

When r and τ are given as functions of s , the preceding results may be applied at once. We shall give a few examples.

A. The circular cylindrical helix is defined by the intrinsic equations

$$r = \text{const.}, \quad \tau = \text{const.}$$

All of its osculants are fixed with reference to the moving trihedral; this is as obvious geometrically as it is analytically. The center of the ruled quadric (46), osculating the surface of binormals, coincides with the center of curvature of the helix; the ruled quadric (51), osculating the surface of principal normals, is an equilateral hyperbolic paraboloid.

The projection of the helix upon its osculating plane at P is symmetric with respect to the principal normal. This is evident geometrically; it is shown analytically by equation (54), from which the term in $\xi\eta$ has vanished, that the focal axis of the osculating conic of this projection coincides with the normal, which is also the axis of aberrancy (58). The conic is an ellipse, parabola, or hyperbola according as $|r/\tau| \leq \sqrt{3}$; that is,⁴ according as the helix cuts the elements of the cylinder upon which it lies at an angle $\leq 30^\circ$. The locus of the centers of these osculating conics as P moves along the helix is a helix on a coaxial cylinder; for $|r/\tau| = \sqrt{3}$, it is a line at infinity.

The null-point of the rectifying plane with respect to the osculating linear complex

¹ Saint-Venant, loc. cit., p. 54; Lilienthal, loc. cit., p. 294.

² Scheffers, loc. cit., I, p. 260.

³ Scheffers, loc. cit., I, p. 197.

⁴ Schell, loc. cit., p. 118.

(63), is at infinity, while the axis of the complex is parallel to the rectifying line. The cylindroid which is the locus of the axes of the penosculating complexes has the rectifying plane as its central plane (69); has the height 2τ (70); and its axis is the normal (72).

The osculating quadric cone (75) is intersected by planes parallel to the normal plane in ellipses, parabolas, or hyperbolas according as $|\tau/r| \leq \sqrt{7/3}$, and the centers of these sections lie in the rectifying plane, by (78); sections parallel to the osculating plane are parabolas with vertices on the bi-normal, by (77); while sections parallel to the rectifying plane are hyperbolas with centers on the principal normal, by (80).

For the seminvariants (84), we find $H > 0$, $G = 0$. Therefore the osculating twisted cubic is a cubical ellipse and the osculating conic of the helix, by § 11, is an ellipse, its center lying upon the normal, by (88).

The property of the circular helix of being a curve of a linear complex¹ is verified by the fact that $\Theta_3 = 0$, by (9).

B. The general cylindrical helix is characterized by the relation $r/\tau = \text{constant}$. The second generator through P of the osculating ruled quadric of the surface of binormals is fixed with respect to the moving trihedral, by (48). The ruled quadric which osculates the surface of principal normals at any point of the helix is an hyperbolic paraboloid, by (52).

C. Curves of constant torsion have certain properties entirely independent of their curvature.

The center of the osculating ruled quadric of the surface of binormals of such curves coincides with the center of curvature; the center of the ruled quadric which osculates the surface of principal normals lies in the rectifying plane, and the diameter through P is perpendicular to the rectifying line.

All of the properties of the osculating linear complex and the cylindroid found for the circular helix hold for curves of constant torsion, except that these configurations are not fixed with respect to the moving trihedral.

The diameter through P of the osculating conic of the curve (88), reduces, by (4), to the form (58) and is therefore coincident with the axis of aberrancy of the projection of the curve upon its osculating plane.

D. Curves of constant curvature exhibit the same properties relative to the osculating conics of their projections upon their osculating planes as do circular helices, except that the conic is not necessarily fixed with reference to the moving trihedral as it is in the case of the helix.

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¹ Staude, loc. cit., p. 155.

